

MOMENTS OF INERTIA AND PRODUCTS OF INERTIA OF A RIGID BODY

Design principles of measurement systems

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Index

<i>Index</i>	<u>1</u>
<i>Notation</i>	<u>4</u>
Chapter 1: Measurement of mass properties: State of the art	<u>5</u>
1.1. Measurement of the mass	<u>6</u>
1.2. Measurement of the centre of mass	<u>6</u>
1.3. Measurement of inertia tensor	<u>7</u>
References	<u>10</u>
Chapter 2: Measurement uncertainty of inertia tensor: classical methods	<u>13</u>
2.1 Measurement uncertainty and error	<u>13</u>
2.2 Measurement uncertainty of the moment of inertia around an axis	<u>14</u>
2.2.1 Torsion pendulum	<u>14</u>
2.2.2 Multifilar pendulum	<u>17</u>
2.3 Full inertia tensor measurement: rotation axes passing through the centre of gravity	<u>19</u>
2.3.1 Definition of the rotation axis	<u>21</u>
2.3.2 Minimal system for the measurement of the inertia tensor	<u>22</u>
2.3.2.1 Condition number of matrix [N]	<u>24</u>
2.3.2.2 Rotation of the reference system	<u>26</u>
2.3.2.3 Error definition	<u>27</u>
2.3.2.4 Uncertainty on the moment of inertia only	<u>28</u>
2.3.2.5 Uncertainty on matrix [N] only	<u>29</u>
2.3.2.6 Sources of measurement uncertainty	<u>29</u>
2.3.2.7 Computation of the measurement uncertainty	<u>31</u>
2.3.2.7.1 Sampling of matrix [N]	<u>32</u>
2.3.2.7.2 Computation of error-propagation coefficients	<u>32</u>
2.3.2.8 Results	<u>32</u>
2.3.2.8.1 Condition number	<u>33</u>
2.3.2.8.2 Practical sets of rotation axes	<u>34</u>
2.3.3 Non minimal systems for the measurement of the inertia tensor	<u>37</u>
2.3.3.1 Results	<u>39</u>
2.4 Full inertia tensor measurement: rotation axes not passing through the centre of gravity	<u>43</u>
2.4.1 Axes passing through a pole	<u>45</u>
2.4.2 Generic system of axes	<u>48</u>
References	<u>49</u>
Chapter 3: Propagation of uncertainty: non classical methods	<u>51</u>
3.1 Equations for the measurement of the inertia tensor	<u>51</u>
3.2 Motion with negligible angular accelerations	<u>54</u>
3.3 Motion with negligible angular velocities	<u>55</u>
3.3.1 Uncertainty definition	<u>57</u>
3.3.2 Sources of uncertainty	<u>57</u>
3.3.2.1 Computation of the measurement uncertainty	<u>58</u>
3.3.2.1.1 Sampling of matrix [N]	<u>58</u>
3.3.2.1.2 Computation of the error propagation coefficients	<u>59</u>
3.3.3 Results	<u>60</u>
3.3.3.1 Condition number	<u>60</u>
3.3.3.2 Error propagation	<u>61</u>
3.3.4 Application: analysis of a vibrating table for the measurement of the inertia tensor components	<u>63</u>

3.4	Generic motion	65
3.4.1	Motion of the system	66
3.4.1.1	Construction of the measurement matrix $[N]$	66
3.4.2	Sources of uncertainty	67
3.4.3	Results	67
3.4.3.1	Condition number	68
3.4.3.2	Error propagation	69
3.4.3.3	Effect of the sampling frequency	71
3.4.3.4	Effect of the frequency content of the motion	73
3.4.3.5	Effect of neglecting angular velocities	75
3.4.3.6	Effect of the statistical distribution of the accelerations along the three axes	77
3.4.4	Analysis of an existing test rig and definition of a quality index	78
3.5	Equations for the measurement of both the centre of gravity location and the inertia tensor	83
3.5.1	Effect of the uncertainty in the location of the centre of gravity on the components of the inertia tensor	85
3.5.2	Modal analysis	87
	References	89
Chapter 4: A test rig for the measurement of the mass properties of rigid bodies		90
4.1	Hardware	90
4.2	Measurement of the mass properties	92
4.2.1	Measurement of the location of the centre of gravity	93
4.2.2	Measurement of the components of the inertia tensor	94
4.3	Instrumentation	95
4.3.1	Measurement of the mass	95
4.3.2	Measurement of the angular displacement of the system	96
4.3.3	Measurement of angular velocities	97
4.3.4	Measurement of accelerations	97
4.3.5	Measurement of the forces acting on the system	98
4.4	Experimental tests	98
4.5	InTenso+ and InTensino+ technical specifications	100
	References	101
Chapter 5: Measurement of the mass properties of bodies with aerodynamic surfaces		102
5.1	Literature survey	102
5.2	Measurement error if non negligible aerodynamic forces are present: Analytical investigation	103
5.3	Uncertainty compensation	105
5.4	Experimental results	108
	References	111
Appendix 1: Definition of mass properties		113
A1.1	Mass	113
A1.2	Centre of mass (Centre of gravity)	113
A1.3	Inertia tensor	114
A1.3.1	Moment of inertia	114
A1.3.2	Inertia tensor definition	115
A1.3.3	Rotation of the inertia tensor	117
A1.3.4	Ellipsoid of inertia	118
A1.3.5	Parallel axis theorem	118

A1.3.6	Conditions for the existence of the inertia tensor	119
A1.3.7	Mohr's circle	121
A1.3.8	Alternative definition of the inertia tensor from the kinetic energy	124
A1.4	Mass properties of systems of rigid bodies	125
A1.5	Inertia tensor in vector notation: inertia vector	126
	References	127
<i>Appendix 2: Short notes of numerical calculus</i>		128
A2.1	Vector and matrix norms	128
A2.1.1	Convergence, norms and inverses	129
A2.2	Effect of a perturbation on a linear system	131
A2.3	Effect of a perturbation on an over-determined linear system	132
	References	135
<i>Appendix 3: Error propagation</i>		136
A3.1	Random variables and statistical distributions	136
A3.1.1	Normal distribution	136
A3.1.2	Beta distribution	137
A3.2	Algebra of random variables	137
A3.1.3	Uncertainty and formula for the propagation of uncertainty	138
A3.3	Condition number and formula for propagation of uncertainty	139
	References	141
<i>Appendix 4: Measurement error and uncertainty definitions</i>		143
A2.4	Measurement error definition	143
A2.5	Measurement uncertainty definition	143
	References	144
<i>Appendix 5: Sampling of the inertia tensors space</i>		145
A5.1	Error definition	145
A5.2	Normalization of the inertia tensor	146
A5.3	Sampling of the normalized inertia tensors (inertia vectors with unitary norm) in diagonal form	147
A5.4	Rotation of the inertia tensors in diagonal form	149
A5.5	Normalization of the moment of inertia around an axis r	150
A5.6	Normalization of the applied torque	150
A5.7	References	152